groups which play such an important rôle in the science of our epoch."* "The group concept was employed in the preceding century (about 1770) simultaneously by Lagrange and Vandermonde, and since this time it occupies a prominent place in the theory of algebraic equations. In regard to this it is only necessary to refer to the name of Galois. Hence group theory has been regarded as a supplement of algebra. This however is incorrect. For the group concept extends far beyond this into almost all other parts of mathematics."†

"It was reserved for Galois to place the theory of equations on a definite foundation by showing that to each equation there corresponds a substitution group in which are exhibited its essential characteristics, and especially those which relate to its solution by other auxiliary equations."‡ "I should reproach myself for forgetting, even in so rapid a resumé, the applications which Lie has made of his theory of groups to the non-Euclidean geometry and to the profound study of the axioms which lie at the basis of our geometric knowledge."§

"The theory of groups, which is making itself felt in nearly every part of higher mathematics, occupies the foremost place among the auxiliary theories which are employed in the most recent function theory."‖ "It need scarcely be added that some modern mathematicians seem to avoid group theory even where it would simplify the treatment of the subject in hand. This seems to be true, for instance, of Hilbert's Grundlagen der Geometrie."¶

LELAND STANFORD UNIVERSITY.

OUR SYMBOL FOR ZERO.

By DR. GEORGE BRUCE HALSTED.

At the Paris International Congress, the paper of my erudite friend of Japanese days, Professor Fiyisawa, attracted, I believe, apart from the great address of Hilbert, more favorable attention than any other.

In praising this paper, I ventured there to emphasize, that the appearance in Japan, before any communication with Europe, of a positional notation for number with precisely the symbol for zero which we now use, and which, as Professor Cajori says in the February number of the AMERICAN MATHEMATICAL MONTHLY, has been supposed of Hindu origin, raised the question for the future historian of mathematics of the relation or connection between these two indubitable, however widely separated, appearances of the same peculiar symbol.

* Picard, *Traite d' analyse*, Vol. 3, 1896, page 492.
† Klein, *Einleitung in die hoehere Geometrie* II, 1893, page 3.
‡ Jordan, *Traite des substitutions*, 1870, Preface.
§ Darboux, *Comptes Rendus*, Vol. 128, 1899, page 528.
‖ Fricke und Klein, *Automorphe Functionen*, Vol. 1, 1897, page 1.
¶ Poincare, *Bulletin des Sciences Mathematiques*, Vol. 26, 1902, page 272.

Professor M. Cantor, who was presiding over the section, seemed perturbed, perhaps at the very idea of future historians of mathematics, and strenuously insisted that the idea of positional notation for number and the symbolization of zero should be accredited to the Babylonians. Of course we know that there are some fragmentary matters which point to the Babylonians using sixty as a number radix, but I have never learned that any Babylonian symbol for zero has been discovered, and the question to which I called attention remained untouched. But now, if we may accept the testimony of Y. Mikami, of Tokio, one of those questions has been answered. He says, "I have found very important relations between the mathematics of India and of China." But we know that Japanese hieroglyphic writing, like Japanese Buddhism, came through China.

Therefore the appearance of our symbol for zero in Hindustan and Japan is not one of the mysterious and seemingly inexplicable coincidences, such as the prehistoric appearance and importance of the cross in the pre-Columbian religions of Yucatan and Mexico. For here we find a water-way connecting Hindustan and Japan through China. But the source of the flow still remains undetermined.

Use of the abacus might have been what suggested the symbolization of zero. But Hindustan from time immemorial used systems of abacal calculation. Greece, and even mathematically stupid Rome, had the abacus.

Every claim for importance in the world of ideas hitherto made for China has evaporated into nothingness. They did not even invent gunpowder or the mariner's compass. Remember their exaltation and subsequent downfall in the history of astronomy.

So it will be as to their claim to our zero symbol. They got it, as they did their Buddhism, from India, and passed on both to Japan.

ANNAPOLIS, MD., *March 7, 1903.*

HARMONIC PAIRS IN THE COMPLEX PLANE.

A PURELY GEOMETRICAL TREATMENT FOR CERTAIN MAPS DEFINED BY THE SUBSTITUTION $w=\frac{1}{2}(z+\frac{1}{z})$.

By ARCHIBALD HENDERSON. Ph. D., Associate Professor of Mathematics, University of North Carolina. Chapel Hill.

§1. A subject of especial interest in the Theory of Functions is the conception of harmonic pairs in the complex plane. The proof given below offers initially a direct interpretation of the equation

$$\left| \frac{z-z_1}{z-z_2} \right| = \rho \ldots\ldots (1),$$

and leads subsequently to the interpretation of the harmonic relation

$$\frac{z_1'-z_1}{z_2-z_1'} \div \frac{z_2'-z_1}{z_2-z_2'} = -1 \ldots . (2),$$

where z, z_1, z_2, z_1', z_2' are complex quantities and ρ is a constant.

In a well-known text-book* the method of interpreting equation (1) seems essentially artificial, although the logic is entirely sound. Instead of interpreting the equation directly, which is the problem set, the writer employs a property of the configuration, wholly unsuspected by one who reads the subject for the first time, which leads, after a series of transformations, to the equation in question.

For the following proof, several lemmas will be introduced.

Lemma 1. If $\frac{z-z_1}{z_2-z} = \kappa$, where κ is real, then z, z_1, and z_2 are collinear. Writing the equation in the form

$$z = \frac{z_1 + \kappa z_2}{1+\kappa}$$

we may say that z divides the stroke from z_1 to z_2 in the ratio $+k : 1$.†

Lemma 2. If $\operatorname{am} \frac{z-z_1}{z-z_2} = a$, then z moves on the arc of a circle through the points z_1 and z_2.‡

Lemma 3. If $\frac{z_1'-z_1}{z_2-z_1'} \div \frac{z_2'-z_1}{z_2-z_2} = \lambda$, where λ is a real quantity, then the four points z_1, z_2, z_1', and z_2' are concyclic.§

This lemma is an immediate consequence of Lemma 2.

Consider now the equation (1) above and its interpretation. It is satisfied *in particular* by the two equations

$$\frac{z-z_1}{z-z_2} = \rho, \quad \frac{z-z_1}{z-z_2} = -\rho,$$

shown by taking absolute values on both sides of these two equations. Hence the locus represented by equation (1) crosses the line through the two points z_1 and z_2 at the two points

$$z_\mu = \frac{z_1 + \rho z_2}{1+\rho}; \quad z_\nu = \frac{z_1 - \rho z_2}{1-\rho},$$

---

* Harkness and Morley's *Introduction to Analytic Functions*.

† Harkness and Morley: *Introduction to Analytic Functions*, §18.

‡ Ibidem, §21.

§ Burkhardt: *Functionen Theorie*, Erster Band, §15, VI.
Harkness and Morley: *Introduction to Analytic Functions*, §25.

which follows from Lemma 2. It is to be noted that since the points z_μ and z_ν divide the stroke from z_1 to z_2 both externally and internally in the same ratio, the pairs z_1 and z_2, z_μ and z_ν are harmonic.

Also $z_c = \frac{1}{2}(z_\mu + z_\nu)$

$$= \frac{1}{2}\left(\frac{z_1+\rho z_2}{1+\rho} + \frac{z_1-\rho z_2}{1-\rho}\right)$$

$$= \frac{z_1-\rho^2 z_2}{1-\rho^2}$$

and this point divides the segment z_1 to z_2 externally in the ratio $1 : \rho^2$.

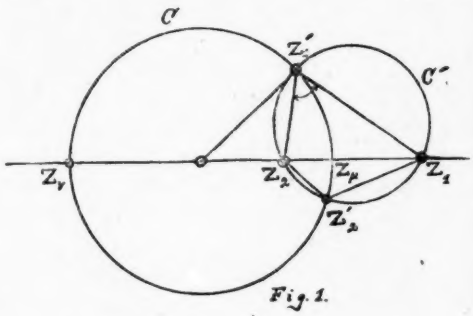


Fig. 1.

Now $\left|\frac{z-z_1}{z-z_2}\right| = \rho$ requires that $\frac{z-z_1}{z-z_2} = \rho(\cos\theta + i\sin\theta)$ and accordingly $z = \frac{z_1 - z_2\rho(\cos\theta+i\sin\theta)}{1-\rho(\cos\theta+i\sin\theta)}$.

Forming now the difference $z - z_c$ we obtain after reduction

$$z - z_c = \frac{\rho(z_1-z_2)}{1-\rho^2}(\cos\theta + i\sin\theta)\left[\frac{1-\rho(\cos\theta-i\sin\theta)}{1-\rho(\cos\theta+i\sin\theta)}\right].$$

Taking absolute values on both sides of this equation and noting that

$$|\cos\theta+i\sin\theta| = 1,$$
$$|1-\rho(\cos\theta+i\sin\theta)| = |1-\rho(\cos\theta-i\sin\theta)|$$

we obtain

$$|z-z_c| = \frac{\rho|z_1-z_2|}{1-\rho^2} = \text{constant},$$

and since z_c is a fixed point, we have the final result, which may be stated as follows:

The equation $\left|\frac{z-z_1}{z-z_2}\right| = \rho$ represents a circle C of radius $\frac{\rho|z_1-z_2|}{1-\rho^2}$ with its center on the line joining the points z_1 and z_2 and dividing the segment from z_1 to z_2 externally in the ratio $1 : \rho^2$.

Now through the points z_1 and z_2 draw any circle C', cutting the circle C in the two points z_1' and z_2'. The equations of the two finite portions of the arc of this circle C', by Lemma 2, are

$$\left.\begin{aligned} \operatorname{am}\frac{z-z_1}{z-z_2} &= \alpha \\ \operatorname{am}\frac{z-z_1}{z-z_2} &= \alpha-\pi \end{aligned}\right\} \ldots\ldots (C').$$

That the circles C and C' are orthogonal can be shown in the following manner. We have the three equations

$$|z_1'-z_c| = \frac{\rho\,|z_1-z_2|}{1-\rho^2},$$

$$|z_1-z_c| = \left|z_1-\frac{z_1-\rho^2 z_2}{1-\rho^2}\right| = \frac{-\rho^2}{1-\rho^2}\,|z_1-z_2|,$$

$$|z_2-z_c| = \left|z_2-\frac{z_1-\rho^2 z_2}{1-\rho^2}\right| = \frac{-1}{1-\rho^2}\,|z_1-z_2|,$$

and therefore,

$$|z_1'-z_c|^2 = |z_1-z_c| \times |z_2-z_c|.$$

That is, the join of the two points z_c and z_1' is a tangent to the circle and hence the circles C and C' are orthogonal.*

Since $\dfrac{|z_1'-z_c|}{|z_2-z_c|} = \dfrac{|z_1-z_c|}{|z_1'-z_c|}$, the triangles $z_c z_2 z_1'$ and $z_c z_1' z_1$ are similar, having the angle $z_2 z_c z_1'$ in common. Putting

$$|z_1-z_c| \times |z_2-z_c| = r^2,$$

the points z_1, z_2 are defined as *inverse* with respect to the circle of center z_c and radius r; and this circle will be said to be drawn *about* any such pair of points.

Another definition of inverse points, found in Harkness and Morley's *Introduction*, based on the orthogonality of the circles C and C', is as follows:

Two points z_1, z_2 are said to be inverse to a circle, when every circle through z_1, z_2 cuts the given circle orthogonally.

§2. The next problem is to interpret the geometric meaning of the equation

$$\frac{z_1'-z_1}{z_2-z_1} \div \frac{z_2'-z_1}{z_2-z_2'} = -1.$$

Since -1 is a real quantity, the four points z_1, z_2, z_1', z_2' are concyclic (Lemma 3). Also clearing out and taking absolute values on both sides, we obtain

* The orthogonality of the two circles may also be found as follows: Isogonality is an invariant property from every linear substitution of the form $w=(az+b)/(cz+d)$. (Forsyth: *Theory of Functions*, §258, page 514). Making the substitution $w=(z-z_1)/(z-z_2)$ in the two equations $|(z-z_1)/(z-z_2)| = \rho$, $\operatorname{am}[(z-z_1)/(z-z^2)]=\alpha$, their maps are $|w| = \rho$, $w=\alpha$,—circle with the origin as center and a half-ray through the origin. Since these figures are orthogonal in the w-plane, their images in the z-plane are also orthogonal.

$$\frac{|z_1'-z_1|}{|z_1'-z_2|}=\frac{|z_2'-z_1|}{|z_2'-z_2|}$$

and hence the points z_1' and z_2' lie on the locus defined by the equation

$$\left|\frac{z-z_1}{z-z_2}\right|=\rho.$$

These four points z_1, z_2, z_1', and z_2' lie on the circle C', two of them, z_1' and z_2', are on C, while two of them, z_1 and z_2, are inverse with respect to C. The pairs z_1 and z_2, z_1' and z_2' are defined as *harmonic pairs* of points.

Since the straight line through z_1 and z_2 may be regarded as a circle of infinite radius, it follows that the pairs z_ν and z_μ, z_1 and z_2 are harmonic pairs on a line, a fact which was proved in a different manner at the beginning of §1.

II.

§3. A very suggestive problem, illustrative of the Riemann surface, is to find the images of certain configurations—straight lines through the origin and concentric circles with origin as center—defined by the equation of correspondence

$$w=\tfrac{1}{2}(z+\frac{1}{z})....(1).$$

The purely geometrical treatment given here is as simple and perhaps simpler than the analytical treatment.*

The equation indicates that the correspondence between the z and the w-planes is not (1, 1), since z takes two distinct values for one of w. The point w is constructed geometrically in the following manner: *Plot the points xepresented by z and $1/z$ and the mid-point of their join gives the point w.*

Consider a circle about the origin, defined by the equation $|z|=r$. If we write $w_1=z$, $w_2=1/z$, and consequently $w=\frac{1}{2}(w_1+w_2)$, we see that as w_1 describes the circle $|z|=r$ in the positive direction, w_2 describes *with the same same angular velocity* the circle $|z|=1/r$ in the opposite direction. This follows from the fact that if we denote

$$\left.\begin{aligned} z&=\rho(\cos\theta+i\sin\theta),\\ \frac{1}{z}&=R(\cos\varphi+i\sin\varphi),\end{aligned}\right\}....(2),$$

the resulting conditions are

$$R=1/\rho,\ \varphi=-\theta....(3).$$

* Holzmueller, *Isogonalen Verwandschaften*, §61, et seq.

If the two circles be described in the sense explained, the point w describes a certain curve which cuts the axes at the points $[\pm\frac{1}{2}(\frac{1}{r}+r), 0][0, \pm(\frac{1}{r}-r)]$ respectively. From the method of description it follows that the curve is symmetrical with respect to the axes. Describe about the origin as center a circle C of radius $\frac{1}{2}(1/r+r)$ and consider for a moment the point M, a point on the locus of the point w, derived from the points F and F' (Fig. 2). The line ML perpendicular to the X-axis is parallel to the line GF, and since ML is parallel to the base of the triangle FGF' and passes through the middle point M of one side FF' of the triangle, therefore it must pass through the middle point N of the other side OF'. Hence it meets the line OF' at the distance $\frac{1}{2}(1/r+r)$ from the center O, and accordingly the point N lies on the circle C. Now

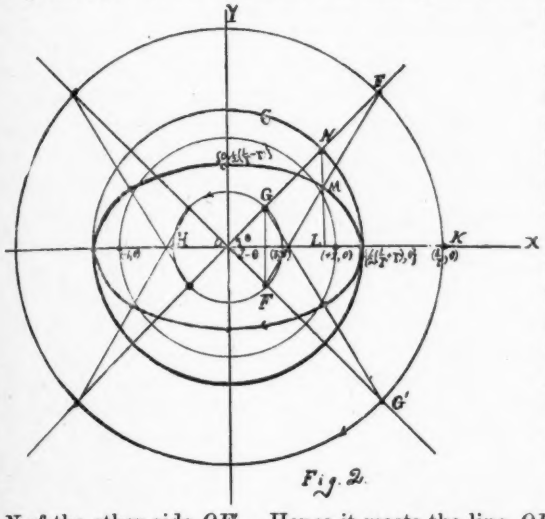


Fig. 2.

$$\frac{NM+ML}{NM}=\frac{NL}{\frac{1}{2}GF}=\frac{ON}{OG}=\frac{\frac{1}{2}(1/r+r)}{r}$$

from which we have

$$NM=\frac{r}{\frac{1}{2}(1/r-r)}ML$$

and substituting we obtain

$$\frac{NL}{ML}=\frac{\frac{1}{2}(1/r+r)}{\frac{1}{2}(1/r-r)}.$$

This proves that the point w describes an ellipse of semi-axes $\frac{1}{2}(1/r+r)$, $\frac{1}{2}(1/r-r)$.* If we have $0<r\leqq 1$, the sense of description is opposite to that of the circle $|z|=r$. If, however, $r=1\ldots\infty$, the ellipse and the circle $|z|=r$ are described in the same sense. Thus when G has reached the point $(0, r)$, G' has reached the point $(0, -\frac{1}{r})$ and M has reached the point $0, \frac{1}{2}(r-\frac{1}{r})$, a point below the X-axis; and so in other cases.

* C. Smith, *Conic Sections*, §121, page 123.

If z describe the circle $KF'G'$, the point w will trace out same ellipse as when the point z described the circle FGH. Since these circles are inverse with respect to the unit circle, the image of which is the segment from $(+1, 0)$ to $(-1, 0)$, we see that the cut for the Riemann surface must be along this segment. The points $(\pm 1, 0)$ are the foci of the ellipses, since $[\frac{1}{2}(\frac{1}{r}+r)]^2-[\frac{1}{2}(\frac{1}{r}-r)]^2 =1$. Any point P on the unit circle, considered as a limiting case of both sets of circles, has accordingly two images placed exactly opposite each other with respect to the segment from $(+1, 0)$ to $(-1, 0)$, a direct consequence of the sense of description detailed above. Hence the portion of the upper sheet (upon which we shall map the region of the z-plane exterior to the unit circle) which is below the horizontal axis is joined to the portion of the lower sheet (upon which we shall map the region of the z-plane interior to the unit circle) above the horizontal axis. The junction is made along the cut from $(+1, 0)$ to $(-1, 0)$. The other portions of the two sheets, upper and lower, are also joined. The whole of each ellipse lies entirely in one or the other of the two sheets, since the points $(\pm 1, 0)$ are the foci of every ellipse.

Consider next the map of a half-ray ${}_0P_\infty$ through the origin (Fig. 3). As the point z moves along the half ray $\theta=\alpha$ from 0 to ∞, the point $1/z$ moves along the half-ray $\varphi=-\alpha$ from ∞ to 0. As $|z|$ ranges from 0 to 1, $\frac{1}{|z|}$ ranges from ∞ to 1; and as $|z|$ ranges from 1 to ∞, $\frac{1}{|z|}$ ranges from 1 to 0. Hence the points z and $1/z$ describe along the two half-rays two projective ranges of points.

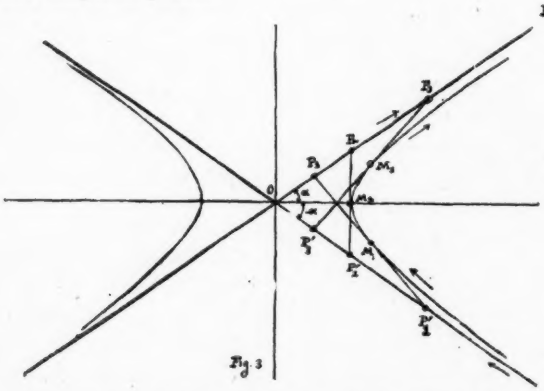


Fig. 3

Since $w=\frac{1}{2}(z+\frac{1}{z})$, the mid-point of any segment PP_1' describes the locus of w. Now the lines PP_1', P_2P_2', ... envelope a conic section which has two points at ∞, one on each half-ray and hence is a hyperbola, to which the half-rays are the asymptotes. When OP_2 is equal to 1, OP_2' is also equal to 1 and M_2 is a vertex of the hyperbola and since OP_2 is distance from the center to the focus, we find that the foci of the hyperbola are $(\pm 1, 0)$. That the system of hyperbolae should be confocal and therefore orthogonal to the former system of ellipses was to be expected, since they were the images of orthogenal loci. The correspondence and sense of description is patent from the drawings, both by means of the lettering and the arrows. For example, let the point z describe

the line EOC beginning at D_∞ in the lower left hand corner. As z moves from D_∞ through E to F, w moves in the upper sheet from D_∞'' through E'' to F''; as

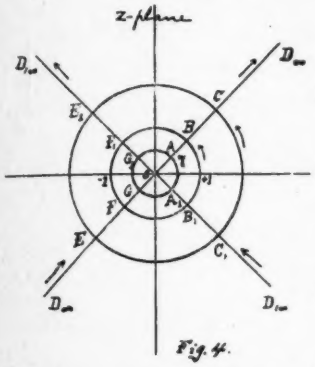


Fig. 4.

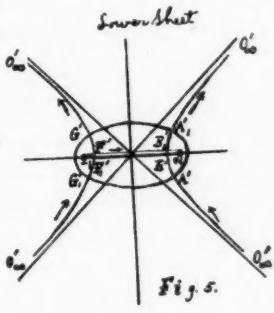


Fig. 5.

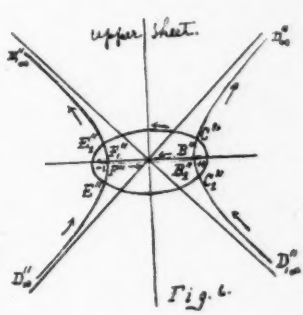


Fig. 6.

z moves from F through G to O, w moves in the lower sheet from F' through G' to O_∞'; as z moves from O through A to B, w moves in the lower sheet from O_∞' lower right hand) through A' to B; and lastly, as z moves from B through C to D_∞, w moves in the upper sheet from B'' through C'' to D_∞''.

THE UNIVERSITY OF CHICAGO, *March, 1903.*

THE GENERAL EUCLIDEAN CONSTRUCTION.

By CHARLES H. SISAM, A. B., Graduate Scholar, Cornell University.

Required to construct, with straight edge and compasses, a length y, given the relation $y=\varphi(a, b, \ldots, l)$; where $a, b, \ldots, l$ represent known lengths and φ is a function involving only rational expressions and square roots.

In order that the construction shall be possible, φ must be homogeneous of first degree. For, let $y_1, a_1, b_1, \ldots, l_1$ be the numerical values $y, a, b, \ldots, l$ for some unit of length z. Then for any other unit of length $(1/k)z$, we must have $ky_1 \equiv \varphi(ka_1, kb_1, \ldots, kl_1)$ for all values of k.

Consider one of the radicals of φ such that the expression under it does not contain any radicals. This expression is composed of a sum of terms of the form $\frac{P}{Q} a^\alpha b^\beta \ldots l^\lambda$; where P, Q, $\alpha, \beta, \ldots, \lambda$ are integers; and, for all the terms of the expression $\alpha+\beta+\ldots+\lambda=c$, a constant, which can—by removing from under the radical sign, if necessary, an even power of any of the quantities $a, b, \ldots, l$—be made equal to 2.

On any line, lay off a convenient length $ON \equiv x$. At N erect a perpendicular and on the perpendicular lay off $NA=a$, $NB=b$, etc. Draw OA, OB, ..., OL.

Let a be positive. On ON lay off $OA_1=a$. At A_1 erect a perpendicular cutting OA in A_1'. By similar triangles, $A_1A_1'=a^2/x$. On ON lay off $OA_2=A_1A_1'$, and erect a perpendicular cutting OA in A_2'. Then $A_2A_2'=a^3/x^2$; and so on, till we obtain $A_{a-1}A'_{a-1}=a^a/x^{a-1}$. If β is positive, lay of, on ON, $OB_1=A_{a-1}A_{a-1}'$ (or equal to NA if $a=1$). Erect a perpendicular cutting OB in B_1'; and so on, till we obtain $B_\beta B_\beta'=\frac{a^a b^\beta}{x^{a+\beta-1}}$. If γ, for example, is negative, draw through B_β' a parallel to ON cutting OC in C_1'. From C_1' drop a perpendicular on ON cutting ON in C_1. Then $OC_1=\frac{a^a b^\beta c^{-1}}{x^{a+\beta-2}}$. On NC lay off $NC_2''=OC_1$. Erect a perpendicular cutting OC_2 in C_2'. Drop a perpendicular C_2C_2' on ON. Then $OC_2=\frac{a^a b^\beta c^{-2}}{x^{a+\beta-3}}$. Continuing in this way, we obtain OL_λ or $L_\lambda L_\lambda'=\frac{a^a b^\beta \ldots l^\lambda}{x^{a+\beta\ldots\lambda-1}}=\frac{a^a b^\beta \ldots l^\lambda}{x}$. By laying off OL_λ in succession P times and applying the known construction for the Q section of a line we obtain $SR=\frac{P}{Q}\frac{a^a b^\beta \ldots l^\lambda}{x}$.

Making a similar construction for each of the terms under the radical considered and taking the algebraic sum of the resulting lines we obtain $TU=\frac{\Sigma\,(P/Q)a^a b^\beta \ldots l^\lambda}{x}$.

From N lay off on ON, $NV=TU$,—away from O if TU is positive. On OV as a diameter construct a circle cutting NA in M (and M'). Then $NM=\sqrt{\Sigma\,\frac{P}{Q}a^a b^\beta \ldots l^\lambda}=m$ (say).

[We may consider $NM'=-NM$. In this way the conjugate values of y may be constructed].

Hence $y=\varphi_1(a, b, \ldots, l, m)$, where φ_1 is homogeneous and contains one less radical than φ. Continuing in this way, we obtain finally $y=\psi(a, \ldots, l, m, \ldots, r)$, where ψ is a sum of terms of the form $(P'/Q')a^{a'}b^{\beta'}\ldots r^{\rho'}$ and $a'+\beta'\ldots+\rho'=1$. Constructing the sum of these terms after the manner above indicat. ed, we obtain the length y required.

A GENERAL THEORY OF PROJECTILES.

By M. E. GRABER, Heidelberg University. Tiffin. Ohio.

There seems to be a lack of uniformity in defining the term *projectile*. Some definitions are too exclusive or special, while others are too inclusive. From a consideration of the conditions involved the following definition seems to answer the purpose well:

A projectile is any body projected from a given point in a given direction subject

to the force of gravity and whatever other forces are inherent in the medium through which the body is projected.

This definition, not being limited to one medium, enables us to apply the term "projectile" not only to particles impelled through the atmosphere but also to particles projected through any medium. This would of course include submarine torpedoes in the category of projectiles.

The method of treatment of the theory of projectiles has generally been first the discussion of motion *in vacuo* and then the discussion of motion in a resisting medium such as the atmosphere. Why not first deduce and discuss formulae of general application to the motion of particles and then consider motion *in vacuo* as a special case with particular values assigned to the different coefficients and factors?

The first consideration in the discussion of the general case is the resistance of the air to the motion of a particle through the atmosphere.

It has been proven by experiment that the resistance of the air encountered by a projectile is proportional to the exposed area. This area is proportional to the square of the diameter of the projectile. Calling this resistance R, we have $R=d^2\phi(v)$, and the corresponding retardation $r=\frac{g}{w}R=g\frac{d^2}{w}\phi(v)$. Putting D (ballistic coefficient) for $\frac{w}{d^2}$, we have $r=\frac{g}{D}\phi(v)$. Assuming that $\phi(v)=B_1v^n$, in which B_1 and n are constants, to be determined by experiment, we get

$$R=\frac{B_1d^2v^n}{g} \text{ and } r=\frac{B_1v^n}{D}....(1).$$

Assuming that the axis of the (oblong) projectile lies constantly in the tangent to the trajectory and that the air is calm and of uniform density, the retardation along the tangent at any point of the trajectory due to air resistance is $\frac{B_1v^n}{D}$; the retardation due to gravity is $g\sin\theta$. Consequently $-\frac{dv}{dt}=\frac{B_1v^n}{D}+g\sin\theta$. The velocities parallel to the X-axis, which is horizontal forwards, and the Y-axis, which is vertical upwards, are $v\cos\theta$ and $v\sin\theta$. The corresponding retardations are $\frac{1}{D}B_1v^n\cos\theta$ and $\frac{1}{D}B_1v^n\sin\theta+g$. Therefore

$$\frac{d(v\cos\theta)}{dt}=-\frac{B_1v^n}{D}\cos\theta....(2),$$

$$\frac{d(v\sin\theta)}{dt}=-\left(\frac{B_1v^n\sin\theta}{D}+g\right)....(3).$$

Performing the indicated differentiations and comparing the resulting equations we get

$$\frac{vd\theta}{dt}=-g\cos\theta....(4),$$

an expression for the resultant of the forces normal to the direction of resistance. Again, represent the horizontal velocity by v_1; then of course $v_1=v\cos\theta$. Substituting this in (2) and (4), we get

$$\frac{dv_1}{dt}=-\frac{B_1 v_1^n}{D\cos^{n-1}\theta}....(5).$$

$$\frac{v_1 d\theta}{dt}=-g\cos^2\theta....(6).$$

The relations between elements of time and elements of the trajectory at any point are given by the following: $dx=v_1 dt$, $dy=v_1\tan\theta dt$, $ds=v_1\sec\theta dt$, where s is the length of the trajectory from the origin. In the above three equations substituting for dt its value $-\frac{v_1}{g}\sec^2\theta d\theta$, we get

$$dx=-\frac{v_1^2}{g}\sec^2\theta d\theta....(7),$$

$$dy=-\frac{v^2}{g}\tan\theta\sec^2\theta d\theta....(8),$$

$$ds=-\frac{v_1^2}{g}\sec^3\theta d\theta....(9).$$

We have now discussed the general theory sufficiently to enable us to derive all the equations of the trajectory *in vacuo*. *In vacuo* the resistance being 0, equation (5) reduces to $dv_1=0$. Consequently v_1 (horizontal velocity) is constant and equal to initial velocity V. Therefore $v\cos\theta=V\cos\phi$. Keeping in mind that ϕ is the initial value of θ, and θ the angle which the tangent of the trajectory at the point x, y makes with the axis of abscissae. Integrating $dt=-\frac{v_1}{g}\sec^2\theta d\theta$ between the limits ϕ and θ, we get

$$t=\frac{V_1}{g}(\tan\phi-\tan\theta)....(10),$$

if $v_1=V_1$. Likewise, by integrating (7), (8), and (9) under the same conditions, we get

$$x=\frac{V_1^2}{g}(\tan\phi-\tan\theta)....(11),$$

$$y=\frac{V_1^2}{2g}(\tan^2\phi-\tan^2\theta)....(12),$$

$$S=\frac{V_1^2}{g}[(\phi)-(\theta)],$$

where $\theta=\int\frac{d\theta}{\cos^3\theta}$ and $\phi=\int\frac{d\phi}{\cos^3\phi}$, both integrals having the same lower limits.

To derive the equation of the trajectory *in vacuo*, eliminate $\tan\theta$ from (11) and (12) by division and addition and we have

$$y=x\tan\phi-\frac{\frac{1}{2}gx^2}{V_1^2},$$

the equation of a parabola whose axis is vertical.

To determine the range, we merely substitute X for x and $-\phi$ for θ in (11); whence $X=\frac{2V_1^2\tan\phi}{g}$. Then since $V\cos\phi=V_1$, $X=\frac{V^2\sin 2\phi}{g}$.

To determine the time of flight, set $\theta=-\phi$ in (10). We have

$$T=\frac{2V_1\tan\phi}{g}=\frac{2V\sin\phi}{g}.$$

To determine velocity, since $V\cos\phi=v\cos\theta=V_1$, we have from (12),

$$v^2\sin^2\theta=V^2\sin^2\phi-2gy.$$

Adding $v^2\cos^2\theta$ to the first member and $V^2\cos^2\phi$ (its equal) to the second member, we get $v^2=V^2-2gy$. In a similar manner, we can determine all the properties of a trajectory *in vacuo*.

A PEDAGOGICAL QUESTION IN SPHERICAL TRIGONOMETRY.

By G. W. GREENWOOD, Professor of Mathematics and Astronomy in McKendree College.

The numerous and seemingly disconnected formulae of right and quadrantal spherical triangles are often found, both by student and instructors, difficult to remember and thus detract from the study of spherical trigonometry.

I have found the application of the general formulae for oblique triangles to these triangles very easy and useful, the necessary sets of formulae being

$$\cos a=\cos b\cos c+\sin b\sin c\cos A....(1),$$

$$-\cos A=\cos B\cos C-\sin B\sin C\cos a....(2),$$

$$\cot a\sin b=\cos b\cos C+\sin C\cot A....(3),$$

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}....(4).$$

The second set can, of course, be written down from the first set by writing $\pi - A$ for a, etc., so that we need remember only the sets (1), (3), and (4). To determine what formula to use in any case, we notice that set (1) connects *three sides and an angle;* set (2), *three angles and a side;* set (3), *two angles and two sides, one of each being included;* set (4), *two angles and two opposite sides.*

We, therefore, choose at once the formula connecting the three given and the one required quantity. In the case of right, or of quadrantal, triangles, one term will always vanish, thus giving the required formula in logarithmic form.

These formulae must sooner or later be learned by the student of spherical trigonometry, and to apply the correct one in any case is easy.

But as an aid in remembering them, we may note that (1) and (3) begin with a *side* and end with the *opposite angle*, the converse being the case in (2). Also in (3) the sides and angles occur in the easily remembered form

$$a,\ b,\ b,\ C,\ C,\ A;$$

and the respective functions occur in the symmetric form

cot, sin, cos, cos, sin, cot.

This form of writing (3) I received from the astronomical lectures of Prof. H. H. Turner, of Oxford, and I have found it a very useful way of remembering it.

I think the deduction of the right and quadrantal triangle formulae from these general formulae is economical, and have found it easier to apply them to right triangles than to use the regular formulae by the means of Napier's rules.

They also apply equally well to the quadrantal triangles which are of such frequent occurence in the usual application of spherical trigonometry to elementary astronomy.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

166. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

If I sell one of my farms for $\$A$,=\$4500, and the other for $\$B$,=\$1800, I will gain $p\%$, =5%, on cost of both; but if I sell the dearer farm for $\$C$,=\$4000, and the other at cost, I will lose $p\%$,=5%. Find the cost of each farm.

Solution by G. B. M. ZERR, A. M., Ph.D., The Temple College, Philadelphia, Pa., and J. E. SANDERS, Hackney, Ohio.

$$\frac{\$A+\$B}{1+p}=\frac{\$6300}{1.05}=\$6000=\text{cost of both farms.}$$

$$\$A+\$B-\frac{\$A+\$B}{1+p}=\frac{\$p(A+B)}{1+p}=\frac{\$6300\times.05}{1.05}=\$300=\text{gain.}$$

$$C+\frac{p(A+B)}{1+p}=\frac{C+p(A+B+C)}{1+p}=\$4000+\$300=\$4300=\text{cost of dearer farm.}$$

$$\frac{A+B}{1+p}-\frac{C+p(A+B+C)}{1+p}=\frac{(A+B)(1-p)}{1+p}-C=\$6000-\$4300=\$1700$$

$$=\text{cost of cheaper farm.}$$

Also solved in a similar manner and with same result by *G. W. GREENWOOD*.

ALGEBRA.

171. Proposed by IDA M. SCHOTTENFELTZ, A. M., New York, N. Y.

$ay^2+a=bxy+cx$, $bx^2+b=axy+cy$. Solve for x and y.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$ay^2+a=bxy+cx$....(1). $bx^2+b=axy+cy$....(2).

From (1), $x=a(y^2+1)/(by+c)$....(3).

(3) in (2) gives $[(a^2+b^2)y^2+2bcy+a^2+c^2](cy-b)=0$.

$$\therefore\ y=b/c,\ y=-\frac{1}{a^2+b^2}\{bc\mp a\sqrt{[-(a^2+b^2+c^2)]}\}.$$

$$x=a/c,\ x=-\frac{1}{a^2+b^2}\{ac\pm b\sqrt{[-(a^2+b^2+c^2)]}\}.$$

Also solved by *MARCUS BAKER*.

GEOMETRY.

193. Proposed by PROFESSOR BEYENS.

Si le rapport du segment d'une base de la sphère à l'hemisphère est m/n, le rapport de l'hauteur du segment à deux bases qui resultera au rayon est égal à $2\sin\frac{1}{3}[\sin^{-1}(n-m)/n]$. [Problem 9699, *Educational Times*.]

Solution by J. E. HITT, Goss, Miss.; G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Penn., and G. W. GREENWOOD, B. A., Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

Let R denote radius of sphere, h the altitude of segment of two bases, $R-h$=altitude of segment of one base. Then, $\pi(R-h)^2[R-\frac{1}{3}(R-h)]/\frac{2}{3}\pi R^3$

$=3(R-h)^2(2R+h)/6R^3=m/n$. Therefore, $(h/R)^3-3(h/R)+2=2m/n$; or, $(h/R)^3-3(h/R)+2(n-m)/n=0$. Applying the proper trigonometric formula, $h/R=2\sqrt{(p/3)}\sin\frac{1}{3}\theta$, where $p=3$, $\theta=\sin^{-1}3q/2p\sqrt{(3/p)}=\sin^{-1}(n-m)/n$, $q=2(n-m)/n$. Hence, $h/R=2\sin\frac{1}{3}[\sin^{-1}(n-m)/n]$.

194. Proposed by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Glass paper weights, having the form of a regular tetrahedron, are to be packed for shipment, each in a paper box. Wanted to know the size and shape of the smallest box for the purpose. How much empty space in each box?

Solution by the PROPOSER.

Shape of box=cube.

Edge of box$=s\sqrt{\frac{1}{2}}$, where s=edge of tetrahedron.

Empty space$=\frac{1}{3}s^3\sqrt{\frac{1}{2}}$.

Occupied space$=\frac{1}{6}s^3\sqrt{\frac{1}{2}}$=volume of tetrahedron.

Total space$=\frac{1}{2}s^3\sqrt{\frac{1}{2}}$=volume of box.

Of the twelve diagonals in the six faces of the box, the six edges of the tetrahedron coincide with one in each face.

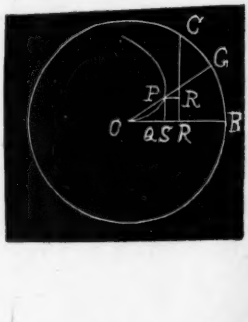
CALCULUS.

160. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A dog at the vertex of a right conical hill pursues a fox at the foot of the hill. How far will the dog run to catch the fox, if the dog runs directly towards the fox at all times, and the fox is continually running around the hill at its foot, the velocity of the dog being 6 feet per second, the velocity of the fox being 5 feet per second, the hill being 100 feet high and 200 feet in diameter at the base?

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let the origin be the vertex of the cone, O the center of the base of the cone, σ=the length of the dog's path, s=the length of the projection of the dog's path on the plane (x, y) or the base of the cone, r=radius vector of this projection, a=100 feet=altitude =radius of base, m=6 feet per second, n=5 feet per second, $n/m=u$, $x^2+y^2=z^2$ is the equation of the cone. Then $u\sigma=a\theta$, where $\theta=\angle COB$, subtended by the fox's path at the center O.



$$d\sigma=(a/u)d\theta=\sqrt{(dx^2+dy^2+dz^2)}=\sqrt{(ds^2+dz^2)}$$

$$=[r^2+(dr/d\theta)^2+(dz/d\theta)^2]^{\frac{1}{2}}\,d\theta.$$

But $r^2=x^2+y^2=z^2$. $\therefore\ dz=dr$.

$\therefore\ d\sigma/d\theta = a/u = [r^2 + 2(dr/d\theta)^2]^{\frac{1}{2}}$, or $d\theta/dr = \left(\frac{2u^2}{a^2 - r^2u^2}\right)^{\frac{1}{2}} = A.$

Let P be the projection of the dog's path on the plane (x, y) at any time; C, the position of the fox at the same time; QC, the tangent at P; $OS = x$; $PS = y$; $OR = t$; $PC = l$; $OP = r$; $\angle POS = \varphi$; $\angle CQS = \psi$. Then $u\sigma = a\theta = a\cos^{-1}(t/a) = a\cos^{-1}[(x + PR')/a]$. But $PR' : CR' = QS : PS$. $\therefore\ PR' = l\cos\psi = l dx/ds$.

$\therefore\ u\sigma = a\theta = a\cos^{-1}[(x + l dx/ds)/a]$ or $a\cos(u\sigma/a) = x + l dx/ds$.

Also $a\cos\theta = x + l dx/ds$. But $a^2 = r^2 + l^2 - 2lr\cos OPC = r^2 + l^2 + 2lr\cos(\varphi - \psi)$, $\cos(\varphi - \psi) = dr/ds$. $\therefore\ a^2 = r^2 + l^2 + 2lr dr/ds$.

$\therefore\ l = \sqrt{[a^2 - r^2 + r^2(dr/ds)^2]} - r dr/ds$.

But $x = r\cos\varphi$, $y = r\sin\varphi$, $ds/dr = \sqrt{[1 + r^2(d\varphi/dr)^2]}$.

$$\frac{dy}{dx} = \frac{\sin\varphi\, dr/d\varphi + r\cos\varphi}{\cos\varphi\, dr/d\varphi - r\sin\varphi}, \qquad \frac{d\varphi}{dr} = \frac{x dy/dx - y}{\sqrt{(x^2 + y^2)(x + y dy/dx)}}.$$

$$\text{and } \frac{ds}{dr} = \frac{\sqrt{\{(x^2 + y^2)[1 + (dy/dx)^2]\}}}{x + y dy/dx}.$$

$$\therefore\ l = \frac{\sqrt{\{a^2[1 + (dy/dx)^2 - (y - x dy/dx)^2]\}} - (x + y dy/dx)}{\sqrt{[1 + (dy/dx)^2]}}.$$

$$\therefore\ a\cos(u\sigma/a) = a\cos\theta = x + \frac{\sqrt{\{a^2[1 + (dy/dx)^2 - (y - x dx/dx)^2]\}} - (x + y dy/dx)}{1 + (dy/dx)^2}.$$

We also have $l^2 = a^2 + r^2 - 2ar\cos(\theta - \varphi)$. This establishes a relation between θ and φ; but it does not simplify the operation any. The differential equation above is too complicated for solution. If we assume that $\theta = \varphi$, and therefore, that the dog is always on a straight line between the fox and the vertex of the hill, the distance he runs can be found; for $d\varphi/dr = d\theta/dr = \frac{x dy/dx - y}{r(x + y dy/dx)} = A$.

$l = a - r$, $x = r\cos\theta = r\cos(u\sigma/a)$, $y = r\sin\theta = r\sin(u\sigma/a)$. Let $u\sigma/a = \beta$.

$$\text{Then } \frac{dy}{dx} = \frac{Arx + y}{x - Ary} = \frac{Ar\cos\beta + \sin\beta}{\cos\beta - Ar\sin\beta}. \qquad dx/ds = \frac{\cos\beta - Ar\sin\beta}{\sqrt{(1 + A^2r^2)}}.$$

Then $a\cos(u\sigma/a) = x + l dx/ds$ becomes

$$a\cos\beta = r\cos\beta + \frac{(a - r)(\cos\beta - Ar\sin\beta)}{\sqrt{(1 + A^2r^2)}}. \qquad \therefore\ \frac{\cos\beta}{\cos\beta - Ar\sin\beta} = \frac{1}{\sqrt{(1 + A^2r^2)}}.$$

$$\therefore\ \tan\beta = \frac{1 - \sqrt{(1 + A^2r^2)}}{Ar} \text{ or } \beta = \tan^{-1}\left(\frac{1 - \sqrt{(1 + A^2r^2)}}{Ar}\right). \text{ Now } \beta = u\sigma/a,$$

$$A=\frac{u\sqrt{2}}{\sqrt{(a^2-r^2u^2)}}. \quad \therefore \sigma=\left[\frac{a}{u}\tan^{-1}\left(\frac{\sqrt{(a^2-r^2u^2)}-\sqrt{(a^2+r^2u^2)}}{ru\sqrt{2}}\right)\right]_{r=0}^{r=a}$$

$$\therefore \sigma=\frac{a}{u}\tan^{-1}\left(\frac{\sqrt{(1-u^2)}-\sqrt{(1+u^2)}}{u\sqrt{2}}\right)=120\tan^{-1}\left(\frac{\sqrt{11}-\sqrt{61}}{5\sqrt{2}}\right)$$

$$=314\tfrac{1}{8} \text{ feet, nearly.}$$

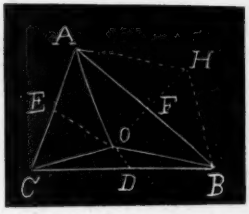
Prof. G. W. Greenwood solves the problem, interpreting it according to Prof. Zerr's assumption, but he gets a slightly different result. The Proposer furnished a partial solution for *The Educational Times*, London, obtaining the same differential equation as that obtained by Prof. Zerr, but by a different method.

MECHANICS.

150. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

O is a point in a plane of a triangle, ABC, and D, E, F are the mid-points of the sides. Show, geometrically, that the system of forces OA, OB, OC is equivalent to the system OD, OE, OF.

Solution by G. W. GREENWOOD, B. A., Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.



Complete the parallelogram $AOBH$, of which OE is one-half the diagonal.

Since the forces OH, OB, are equivalent to OH, $\frac{1}{2}OA$ and $\frac{1}{2}OB$ will be equivalent to OF.

Similarly, $\frac{1}{2}OB$ and $\frac{1}{2}OC$ will be equivalent to OD, and $\frac{1}{2}OC$ and $\frac{1}{2}OA$ will be equivalent to OE.

Hence the system OA, OB, OC is equivalent to the system OD, OE, OF.

151. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

An elastic ball is projected along a horizontal tube, striking first the bottom, then the top, then the bottom and so on. Find the number of times the ball will strike the top.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let v=velocity just before first impact, r=diameter of tube, $v_1=\sqrt{(2gr)}$, e=coefficient of restitution. Then velocity of leaving bottom=ev, velocity of arrival at top=$ev-v_1$, velocity of arrival at bottom=$e^2v-ev_1+v_1$, velocity of second arrival at top=$e^3v-e^2v_1+ev_1-v_1$, velocity of third arrival at top=$e^5v-e^4v_1+e^3v_1-e^2v_1+ev_1-v_1$.

$\therefore$ Velocity of nth arrival at top=$e^{2n-1}v-v_1(e^{2n-2}-e^{2n-3}+e^{2n-4}-e^{2n-5}+\ldots.+e^4-e^3+e^2-e+1)$, and this velocity=$v_1$.

$\therefore e^{2n-1}v-v_1(e^{2n-2}+e^{2n-4}+\ldots.+e^4+e^2)+v_1(e^{2n-3}+e^{2n-5}+\ldots.+e^3+e)=2v_1$.

$\therefore e^{2n-1}v - e^2 v_1\left(\frac{e^{2n-2}-1}{e^2-1}\right) + ev_1\left(\frac{e^{2n-2}-1}{e^2-1}\right) = 2v_1.$

$\therefore v(e^2-1)e^{2n-1} - ev_1 e^{2n-1} + v_1 e^{2n-1} = v_1(e^2+e-2).$

$\therefore e^{2n-1} = \frac{v_1(e+2)}{v(e+1)-v_1} = A$, suppose.

$\therefore 2n-1 = \log A/\log e.$ $\therefore n = \frac{1}{2}\log(Ae)/\log e =$ the number required.

DIOPHANTINE ANALYSIS.

107. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Required the least three positive integral numbers such that the sum of all three of them, and the sum of every two of them shall be a square number.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let x, y, and z = the numbers.

Then $x+y = \square = h^2$, $x+z = \square = k^2$, $y+z = \square = l^2$.

$\therefore x+y+z = \frac{1}{2}(h^2+k^2+l^2) = \square = s^2 \ldots (1).$

$\therefore x = s^2 - l^2$, $y = s^2 - k^2$, and $z = s^2 - h^2$.

Put $l = s-m$, $k = s-n$, and $h = s-r$. Substituting these values in (1), we obtain $3s^2 - 2s(m+n+r) + m^2 + n^2 + r^2 = 2s^2$.

Solving for s, we find $s = m+n+r \pm \sqrt{[2(mn+mr+nr)]}$.

Take $mn+mr+nr = mn+r(m+n) = 2b^2$.

$\therefore s = m+n+x \pm 2b,$

$x = 2ms - m^2 = m(2s-m) \ldots (2),$

$y = 2ns - n^2 = n(2s-n) \ldots (3),$

$z = 2rs - r^2 = r(2s-r) \ldots (4).$

Put $n = m+a$. Then $mn+r(m+n) = m(m+a)+r(2m+a) = 2b^2$.

$\therefore r = \frac{2b^2 - m(m+a)}{2m+a}$, and $s = \frac{(m+a)(3m+a) - am + 2b[b \pm (2m+a)]}{2m+a}$.

Substituting in (2), (3), and (4), and multiplying by $(2m+a)^2$, we obtain the following general values:

$$x = m(2m+a)\{2(m+a)(2m+a) - am + 4b[b \pm (2m+a)]\},$$
$$y = (m+a)(2m+a)\{(2m+a)^2 - am + 4b[b \pm (2m+a)]\},$$
$$z = [2b^2 - m(m+a)]\{2[b \pm (2m+a)]^2 - m(m+a)\},$$
$$x+y = \{(2m+a)[2b \pm (2m+a)]\}^2,$$
$$x+z = \{m^2 + 2b[b \pm (2m+a)]\}^2,$$
$$y+z = \{(m+a)^2 + 2b[b \pm (2m+a)]\}^2,$$
$$x+y+z = \{(m+a)(3m+a) - am + 2b[b \pm (2m+a)]\}^2.$$

For *positive* values, we have the general condition, $2b^2 > m(m+a)$; also, when $b-(2m+a)$ is used, the condition, $b > 2m+a$.

When x, y, and z have a common divisor, lowest values are obtained by dividing by the highest common *square* factor.

Multiple values may be obtained by multiplying any set of values of x, y, and z by a *square* number.

m, a, and b may be any integers, subject to the conditions for positive values.

$a=0$, when $m=n$.

The least values are obtained by taking $m=a=1$, and $b=2$, and dividing by 3^2, the highest common square factor.

Whence $x=17$, $y=32$, $z=32$.

These values may also be found by taking $a=0$, and $m=b=1$. Then $x=32$, $y=32$, $z=17$.

The least *different* values are obtained by taking $m=a=1$, and $b=4$, using $b+2m+a$, and dividing by 3^2.

Whence $x=41$, $y=80$, $z=320$.

By using $b-(2m+a)$, in the last case, we find $x=9$, $y=16$, $z=0$.

Excellent solutions of this problem were received from *PROFESSORS ZERR*, *CROSS*, and *WALKER*, and the late *JOSIAH H. DRUMMOND*. Mr. Cross sent in two solutions, one of which was a solution of the generalized problem. If space permits, his solution of the generalized problem will be published in the next issue of the MONTHLY.

Professor Walker should have been credited with a solution of problem 106.

No solutions of problems 105 and 108 have yet been received.

AVERAGE AND PROBABILITY.

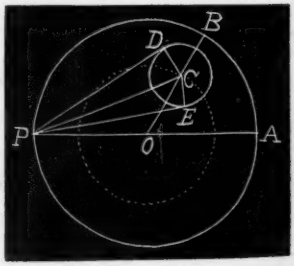
90. Proposed by WALTER H. DRANE, Graduate Student, Harvard University.

During a rain-storm a circular pond is formed in a circular field. If a man undertakes to cross the field in the dark, what is the chance that he will walk into the pond?

III. Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

In the following solution, we assume that the path of the man is along a random straight line drawn from a random point in the circumference of the field. We also assume that the number of favorable paths is to the total number of paths as the arc of a circle (radius, the line drawn from the random point on the circumference of the field to the center of the pond) intercepted by the pond, is to the semi-circumference of the same circle, and that all directions of the path are equally probable; that all values of the radius of the pond less than the radius of the field are also equally probable; that all points on the circumference of the field are equally likely to become the point of starting across the field; and that all points of the field are equally likely to become the center of the pond.

Let O be the center of the field, radius $AO=R$; C, the center of the pond; and P, the point where the man enters the field.



Let $x=OC$, the distance from the center of the field to the center of the pond; $z=CD=CE$, the radius of the pond; $\theta=\angle AOB$; and $\phi=\angle CPE=\angle CPD=\sin^{-1}\left(\frac{z}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)$

Then, (1) the chance that the center of the pond lies on the area comprised between two

concentric circles whose common center is O and whose radii are x and $x+dx$ is $C_1=\frac{2\pi x dx}{\pi R^2}=\frac{2xdx}{R^2}$; (2) the chance that the radius of the pond lies between z and $z+dz$ is $C_2=\frac{dz}{R-x}$; (3) the chance that the line of centers, OC, makes an angle with the diameter, AP, between θ and $\theta+d\theta$ is $C_3=d\theta/\pi$; (4) the chance that the path of the man lies between ϕ and $\phi+d\phi$ is $C_4=d\phi/\pi$.

The chance of the concurrence of all of these events is $C_0=C_1C_2C_3C_4$, and the chance of the concurrence of all these events for all values of the variables is

$$C=\int C_1\int C_2\int C_3\int C_4$$

The limits of θ are 0 and π and doubled; the limits of x are 0 and R; the limits of z are 0 and $R-x$; and the limits of ϕ are 0 and ϕ.

$$\therefore\ C=\int_0^{\pi}\frac{d\theta}{\pi}\int_0^{R}\frac{2xdx}{R^2}\int_0^{R-x}\frac{dz}{R-x}\int_0^{\phi}d\phi=\frac{2}{\pi^2R^2}\int_0^{\pi}d\theta\int_0^{R}xdx\int_0^{R-x}\frac{dz}{R-x}\phi$$

$$=\frac{2}{\pi^2R^2}\int_0^{\pi}d\theta\int_0^{R}xdx\int_0^{R-x}\frac{1}{R-x}\sin^{-1}\left(\frac{z}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)dz$$

$$=\frac{2}{\pi^2R^2}\int_0^{\pi}d\theta\int_0^{R}\frac{xdx}{R-x}\left[z\sin^{-1}\left(\frac{z}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)+\sqrt{R^2+x^2+2Rx\cos\theta}\right]_0^{R-x}$$

$$=\frac{2}{\pi^2R^2}\int_0^{\pi}d\theta\int_0^{R}[x\tan^{-1}\left(\frac{R-x}{2\sqrt{(Rx)}\cos\frac{1}{2}\theta}\right)+2\sqrt{(Rx)}\cos\tfrac{1}{2}\theta$$

$$-\frac{x}{R-x}\sqrt{(R^2+x^2+2Rx\cos\theta)}]\,dx=\frac{2}{\pi^2R^2}\int_0^{\pi}d\theta\left[\tfrac{1}{2}x^2\tan^{-1}\left(\frac{R-x}{2\sqrt{(Rx)}\cos\frac{1}{2}\theta}\right)\right]_0^{R}d\theta$$

$$+\tfrac{1}{2}\int_0^{R}dx\int_0^{\pi}-\frac{x(R+x)\sqrt{(Rx)}\cos\frac{1}{2}\theta}{R^2+x^2+2Rx\cos\theta}d\theta+2\sqrt{R}\int_0^{R}\frac{x^{\frac{1}{2}}dx}{R-x}\int_0^{\pi}\cos\tfrac{1}{2}\theta d\theta$$

$$-\int_0^{\pi}d\theta\int_0^{R}\frac{x}{R-x}\sqrt{(R^2+x^2+2Rx\cos\theta)}dx=\frac{2}{\pi^2R^2}\left\{\int_0^{R}x\log\left(\frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\right)dx\right.$$

$$\left.+4\sqrt{R}\int_0^{R}\frac{x^{\frac{1}{2}}dx}{R-x}-\int_0^{\pi}d\theta\int_0^{R}\frac{x}{R-x}\sqrt{(R^2+x^2+2Rx\cos\theta)}dx\right\}$$

$$=\frac{2}{\pi^2R^2}\left\{\tfrac{1}{2}x^2\log\frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\right]_{x=R}-\tfrac{1}{2}\sqrt{R}\int_0^{R}\frac{x^{\frac{1}{2}}dx}{R-x}+4\sqrt{R}\int_0^{R}\frac{x^{\frac{1}{2}}dx}{R-x}$$

$$-\int_0^\pi d\theta \int_0^R \frac{x}{R-x}\sqrt{(R^2+x^2+2Rx\cos\theta)}\,dx \Big\} = \frac{2}{\pi^2 R^2}\Big\{ \tfrac{1}{2}x^2 \log\frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\Big]_{x=R}$$

$$+\frac{z}{2}\sqrt{R}\Big[-\tfrac{2}{3}x^{\frac{3}{2}} - 2Rx^{\frac{1}{2}} + R^{\frac{3}{2}}\log\Big(\frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\Big)\Big]\Big]_0^R$$

$$-\int_0^\pi d\theta \int_0^R \frac{x}{R-x}\sqrt{(R^2+x^2+2Rx\cos\theta)}\,dx\Big\}$$

$$= \frac{2}{\pi^2 R^2}\Big\{ -\tfrac{2}{3}\tfrac{8}{3}R^2 + \Big[\tfrac{1}{2}(x^2+7R^2)\log\frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\Big]_{x=R}$$

$$-\int_0^\pi d\theta \int_0^R \frac{x}{R-x}\sqrt{(R^2+x^2+2Rx\cos\theta)}\,dx\Big\}$$

$$= \frac{2}{\pi^2 R^2}\Big\{ -\tfrac{2}{3}\tfrac{8}{3}R^2 + \Big[\tfrac{1}{4}(x^2+7R^2)\log\Big(\frac{\sqrt{R}+\sqrt{x}}{\sqrt{R}-\sqrt{x}}\Big)\Big]_{x=R}\int_0^\pi \cos(\tfrac{1}{2}\theta)d\theta$$

$$+\int_0^\pi d\theta\Big[\tfrac{1}{2}(x+R\cos\theta)\sqrt{(R^2+x^2+2Rx\cos\theta)}$$

$$+\tfrac{1}{2}R^2\sin^2\theta\log[\sqrt{(R^2+x^2+2Rx\cos\theta)}+x+R\cos\theta]+R\sqrt{(R^2+x^2+2Rx\cos\theta)}$$

$$-2R^2\cos^2\tfrac{1}{2}\theta\log[\sqrt{(R^2+x^2+2Rx\cos\theta)}+R-x-2R\cos^2\tfrac{1}{2}\theta]-2R^2\cos\tfrac{1}{2}\theta\times$$

$$\log\Big(\frac{\sqrt{(R^2+x^2+2Rx\cos\theta)}+2R\cos\frac{1}{2}\theta)}{R-x}-\cos\tfrac{1}{2}\theta\Big)\Big]_0^R$$

$$= \frac{2}{\pi^2 R^2}\Big[-\tfrac{2}{3}\tfrac{8}{3}R^2 + R^2\int_0^\pi (2\cos^2\tfrac{1}{2}\theta - \tfrac{1}{2}\cos\theta + 2\cos\tfrac{1}{2}\theta - 1)d\theta$$

$$+R^2\int_0^\pi (\tfrac{1}{2}\sin^2\theta + 2\cos^2\tfrac{1}{2}\theta + 2\cos\tfrac{1}{2}\theta)\log\Big(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)d\theta\Big]$$

$$= \frac{2}{\pi^2}\Big\{-\tfrac{8}{3}-\pi+\Big[(-\tfrac{1}{4}\sin\theta\cos\theta+\sin\theta+4\sin\tfrac{1}{2}\theta+\tfrac{5}{4}\theta)\log\Big(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)\Big]_0^\pi$$

$$-\int_0^\pi(-\tfrac{1}{4}\sin\theta\cos\theta+\sin\theta+4\sin\tfrac{1}{2}\theta+\tfrac{5}{4}\theta)\frac{\sin(\frac{1}{2}\theta)d\theta}{2\cos\frac{1}{2}\theta(1+\cos\frac{1}{2}\theta)}\Big\},$$

$$= \frac{2}{\pi^2}\Big\{-\tfrac{8}{3}-\pi+\Big[(4+\tfrac{5}{4}\theta)\log\Big(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)\Big]_{\theta=\pi}+\tfrac{1}{4}\int_0^\pi(1-\cos\tfrac{1}{2}\theta)\cos\theta d\theta$$

$$-\int_0^\pi(1-\cos\tfrac{1}{2}\theta)d\theta-2\int_0^\pi\Big(\frac{1-\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)d\theta-\tfrac{5}{4}\int_0^\pi\frac{\theta(1-\cos\frac{1}{2}\theta)d\theta}{\sin\theta}\Big\},$$

$$=\frac{2}{\pi^2}\Big[-\tfrac{8}{3}-\pi+\Big[(4+\tfrac{5}{4}\theta)\log\Big(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)\Big]_{\theta=\pi}-\tfrac{1}{6}-(\pi-2)-[4\log\Big(\frac{1+\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\Big)$$

$$-2\theta\Big]_{\theta=\pi}-\tfrac{5}{4}\Big[\theta\log\tan\tfrac{1}{2}\theta\Big]_0^{\pi}-\tfrac{5}{4}\int_0^{\pi}\log\tan(\tfrac{1}{2}\theta)d\theta^{*}+\tfrac{5}{4}\Big[\theta\log\tan\tfrac{1}{4}\theta\Big]_0^{\pi}$$

$$-\tfrac{5}{4}\int_0^{\pi}\log\tan(\tfrac{1}{4}\theta)d\theta\Big\}=\frac{2}{\pi^2}\Big(-\tfrac{5}{6}-4\log 2-\tfrac{5}{4}\int_0^{\pi}\log\tan(\tfrac{1}{4}\theta)d\theta\Big)=\frac{2}{\pi^2}\Big(-\tfrac{5}{6}-4\log 2$$

$$-5\int_0^{\frac{1}{4}\pi}\log\tan\theta d\theta\Big)=\frac{2}{\pi^2}\Big(40\sum_{n=1}^{n=\infty}\frac{2n-1}{(4n-3)^2\,(4n-1)^2}-\tfrac{5}{6}-4\log 2\Big)$$

$$=\frac{2}{\pi^2}(40\times.114488335\dagger-\tfrac{5}{6}-4\log 2)=\frac{2}{\pi^2}(4.5795334-\tfrac{5}{6}-4\log 2).$$

NOTE.—The above solution of the "pond problem" was attempted at the same time that the published solution of this problem was prepared, but owing to an error in finding an approximate value of

$$\int_0^{\frac{1}{4}\pi}\log\tan\theta d\theta$$

the result obtained appeared to be negative. The solution of problem 156, Calculus Department, involved the evaluation of this same definite integral and in finding the approximate value of this integral, again the result was put in the former answer of the "pond problem" and thus led to a consistent answer.

At the time the former solutions of this problem were published it was my intention to publish this solution, for the reason that about thirty years ago, a controversy arose between Mr. Henry Heaton, of Atlantic, Iowa, and Dr. Joel E. Hendricks, Editor of the *Analyst*, concerning the method of solution of problem 135, of the *Analyst*, this problem involving the same point of difference of solution as is involved in the above solution and my former solution.

In this controversy, Mr. Heaton championed the method followed out in the above solution, and Dr. Hendricks defended the method of the two former published solutions. Mr. Heaton tells us that Professor E. B. Seitz was first with Dr. Hendricks, but was finally convinced by the argument of Mr. Heaton conveyed to him through correspondence. Professor Seitz then converted Dr. Hendricks.

The controversy was again renewed when in 1877, Mr. Heaton sent to Dr. Artemas Martin, now of Washington, D. C., a solution of problem 27, *Mathematical Visitor*, a journal edited and published by Dr. Martin himself,—this problem involving the same bone of contention. Mr. Heaton says, "When I sent Mr. Martin my solution, he wrote back that it was wrong, and said that he had a correct solution from another contributor, and that Professor Benjamin Pierce, and Professor Woolhouse, of England, had decided my solution wrong. I immediately wrote Professor Pierce inclosing Mr. Martin's letter and my solution." In reply to this letter, Mr. Heaton received the following letter from Professor Pierce, the original of which Mr. Heaton kindly loaned me:

MY DEAR SIR:

Mr. Martin informed you correctly concerning my decision. But upon reconsideration, I have decided that I must reverse it, and I now regard your solution as correct. I shall write the grounds of my reversal to Mr. Martin.

Yours faithfully,

BENJAMIN PIERCE.

Thus the controversy ended, and many problems in probability have been solved indifferently by one method or the other from that time to the present.

The solution of the "pond problem" again raises the question as to which method is correct. Certainly, both methods can not be correct, since they lead to different results, when the assumptions as to the distribution of the several events are granted in any particular way.

In the first place, the problem is indefinite, since as many different solutions may be obtained as there are ways of interpreting it and assigning the laws of distribution of the several events. When these have once been assumed then all solutions by whatever method should lead to the same result.

Suppose that the distribution of the events are assumed as in the solution above. Then the solution

*See problem 123, Calculus Department.

†This is the value computed by Professor Zerr, problem 156, Calculus Department.

of the problem is based upon the definition of the probability of an event which is the *total number of favorable cases divided by the total number of cases.*

In this problem, the favorable cases have been assumed to be proportional to the length of the arc $\phi \times PE$ and the total number of cases proportional to the length of the semi-circumference $\pi \times PE$.

Hence, the chance that the man crosses the pond for any particular value of ϕ, x and θ being constant, is

$$C_1 = \frac{\phi \times PE}{\pi \times PE} = \phi/\pi = \frac{\sin^{-1}[z \div (R^2 + 2x^2 + 2Rx\cos\theta)]}{\pi} = \frac{f(z)}{\pi}.$$

Now, all values of $f(z)$ between the limits $z=0$ and $z=R-x$ must be considered. Let

$$h = \frac{0+R-x}{n} = \frac{R-x}{n}. \quad \text{Then,} \quad \sum_{z=0}^{z=R-x} f(z) = f(h)+f(2h)+f(3h)+\ldots+f(nh).$$

The chance that $f(z)$ has the value of any particular term as $f(kh)$ of this series is $1/n$, or $h/(R-x)$.

Hence, the chance that the man crosses the pond is the product of the chance that $f(kh)$ is the value of $f(z)$ multiplied by $f(kh)/\pi = f(kh)/[\pi(R-x)]$.

Hence, the chance that the man crosses the pond for all values of $f(z)$ between the limits $z=0$ and $z=R-x$, x and θ being constant, is

$$C_2 = \frac{f(h)+f(2h)h+f(3h)h+\ldots+f(nh)h}{\pi(R-x)} = \frac{\int_0^{R-x} f(z)dz}{\pi(R-x)} = F(x).$$

$F(x)$ is the chance of the man crossing the pond when the center of the pond is at a certain fixed point. But the center of the pond may be any point in the field. The chance that the center of the pond falls on any point between two concentric circles whose radii are x and $x+dx$ is $\pi x dx/\pi R^2 = 2x dx/R^2$.

Hence, the chance of the man crossing the pond is $(2x dx/R^2)F(x)$, and for all values of x between $x=0$ to $x=R$ the chance is

$$C_3 = \int_0^R \frac{2x dx}{R^2} F(x) = \int_0^R \frac{2x dx}{R^2} \int_0^{R-x} \frac{f(z)dz}{\pi(R-x)} = \Psi(\theta).$$

The chance that θ has any value between θ and $\theta+d\theta$ is $d\theta/\pi$.

Hence, the chance that the man crosses the pond for all values of $\psi(\theta)$ between $\theta=0$ to θ is

$$C_4 = \int_0^\pi \frac{d\theta}{\pi} \Psi(\theta) = \int_0^\pi \frac{d\theta}{\pi} \int_0^R \frac{2x dx}{R^2} F(x) = \int_0^\pi \frac{d\theta}{\pi} \int_0^R \frac{2x dx}{R^2} \int_0^{R-x} \frac{f(z)dz}{\pi(R-x)}.$$

Thus it appears that the first method when made to obey strictly the definition of probability leads to the same form for computation as the second. The first two solutions of this problem as published in the Decemben Number, of Vol. VII, of the MONTHLY, are, therefore, incorrect.

Dr. Martin informs me that Professor Seitz sent him a solution of this problem and obtained the same result as that given in the two solutions previously published. Professor Seitz, in his day, solved a great many very difficult probability problems, the solutions of very few of which have since been found erronous, if the laws of distribution assumed by him be grant·d.

The above solution of the "pond problem" has the sanction of the ablest living mathematicians in this country.

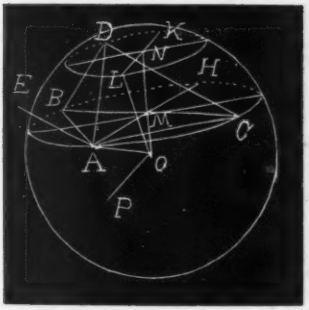
This point, viz., that the total number of events are not to be obtained in such problems by writing for the denominator a certain set of definite integrals, should not be overlooked in solving probability problems of this particular in the future.

130. Proposed by LON C. WALKER. A. M., Graduate Student, Leland Stanford University, Cal.

Four points are taken at random on the surface of a given sphere; show that the average volume of a tetrahedron formed by the planes passing through the the points taken three and three, is 1-35 of the volume of the given sphere.

I. Solution by the PROPOSER.

Choose A, B, C, D as the four random points; O the center of the given sphere with radius r; $ABFE$ a great circle through A, B; ABC a small circle through A, B, C with center S; DGF a small circle through D parallel to ABC with center P; M the middle point of AB.



Put $OP=x$, $AS=r_1$, $\angle AOB=\theta$, $\angle OMS=\phi$, $\angle CAB=\psi$, $\angle SAM=\psi_1$. Then we have

$AM=r\sin\frac{1}{2}\theta=r_1\cos\psi_1$,

$SM=r\cos\frac{1}{2}\theta\cos\phi=r_1\sin\psi_1$,

$OS=r\cos\frac{1}{2}\theta\sin\phi$,

$AC=2r_1\cos(\psi-\psi_1)$,

$PD=\sqrt{(r^2-x^2)}$,

$r_1=r(\sin^2\frac{1}{2}\theta+\cos^2\frac{1}{2}\theta\cos^2\phi)^{\frac{1}{2}}$, area $ABC=2rr_1\sin\frac{1}{2}\theta\sin\psi\cos(\psi-\psi_1)$,

volume of tetrahedron $D-ABC=\frac{1}{3}SP.\text{area}ABC=\frac{2}{3}rr_1(x+r\cos\frac{1}{2}\theta\sin\phi)\sin\frac{1}{2}\theta\times\sin\theta\cos(\psi-\psi_1)$. Hence we have for the required average volume

$$V=\frac{1}{(4\pi r^2)^3}\int_0^\pi\int_0^\pi\Big[\int_0^{\frac{1}{2}\pi+\psi_1}\Big(\int_{-r\cos\frac{1}{2}\theta\sin\phi}^{r}\tfrac{2}{3}rr_1(x+r\cos\tfrac{1}{2}\theta\sin\varphi)\sin\tfrac{1}{2}\theta\sin\psi$$

$$\cos(\psi-\psi_1).2\pi r dx$$

$$+\int_{r\cos\frac{1}{2}\theta\sin\phi}^{r}\tfrac{2}{3}rr_1(x-r\cos\tfrac{1}{2}\theta\sin\phi)\sin\tfrac{1}{2}\theta\sin\psi\cos(\psi-\psi_1).2\pi r dx\Big)\,4rr_1\sin\psi\cos(\psi-\psi_1)d\psi$$

$$+\int_0^{\frac{1}{2}\pi-\psi_1}\Big(\int_{-r\cos\frac{1}{2}\theta\sin\phi}^{r}\tfrac{2}{3}rr_1(x+r\text{cys}\tfrac{1}{2}\theta\sin\phi)\sin\tfrac{1}{2}\theta\sin\psi\cos(\psi+\psi_1).2\pi r dx$$

$$+\int_{r\cos\frac{1}{2}\theta\sin\phi}^{r}\tfrac{2}{3}rr_1(x-r\cos\tfrac{1}{2}\theta\sin\phi)\sin\tfrac{1}{2}\theta\sin\psi\cos(\psi+\psi_1).2\pi r dx\Big)4rr_1\sin\psi\cos(\psi+\psi_1)\Big]d\phi$$

$$\times 2\pi r^2\sin\theta d\theta$$

$$=\frac{r}{3}\int_0^\pi\int_0^\pi\Big[\int_0^{\frac{1}{2}\pi+\psi_1}r_1\sin^2\psi\cos^2(\psi-\psi_1)d\psi+\int_0^{\frac{1}{2}\pi+\psi_1}r_1^2\sin^2\psi\cos^2(\psi+\psi_1)d\psi\Big]$$

$$\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta(1+\cos^2\tfrac{1}{2}\theta\sin^2\phi)d\theta d\phi$$

$$=\frac{r^3}{24}\int_0^\pi\int_0^\pi(\sin^2\tfrac{1}{2}\theta+3\cos^2\tfrac{1}{2}\theta\cos^2\phi)(1+\cos^2\tfrac{1}{2}\theta\sin^2\phi)\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta d\theta d\phi$$

$$=\frac{r^3}{24}\int_0^\pi(\tfrac{3}{35}\tfrac{2}{5}+\tfrac{1}{35}\cos2\phi-\tfrac{2}{35}\cos4\phi)d\phi=\tfrac{4}{105}\pi r^3=\tfrac{1}{35}$$ of the volume of given sphere.

MISCELLANEOUS.

130. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

From a cloud of angular elevation $\phi=45°$, a streak of lightning darted to the earth. The temperature of the atmosphere was $t=80°$, and the percentage of humidity $p=90$. After $m=3$ seconds, the report of the stroke at the earth was heard. How far away from the observer did the streak of lightning (1) start, and (2) strike the earth?

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let H=tension of aqueous vapor at $t°=80°$.

Let H'=tension of aqueous vapor at dew point.

Then $(H'/H)100=p=90$, $80°F=26°.7C$.

Now $H=26.045$mm. $\therefore$ $H'=\frac{9}{10}$ of 26.045mm.$=23.4405$mm.

Let us take the barometric pressure at 740mm.$=P$.

Then $740\text{mm.}-23.4405\text{mm.}=716.5)95$mm. will represent the pressure of dry air.

Let W=weight of litre of dry air at $80°F$.

W'=weight of vapor in one litre.

$$W=1.293187\times\frac{716.5595}{760}\times\frac{1}{1+.00367\times26.7}=1.110667 \text{ grams.}$$

$$W'=1.293187\times.6235\times\frac{23.4405}{760}\times\frac{1}{1+.00367\times26.7}=.022653 \text{ grams.}$$

$W+W'=$=1.13332 grams=total weight of litre.

$V=\sqrt{(Pk/\rho)}$, where $k=1.41$=ratio of specific heat of air at constant pressure to the specific heat at constant volume.

$P=740$mm.$=986634$ dynes per square cm., ρ=density$=.00113332$.

$\therefore$ $V=\sqrt{[(986634\times1.41)/.00113332]}=350.36$ meters per second.

$3\times60\times350.36=63.0648$ kilometers=distance of observer from point on earth where lightning strikes. If we regard the distance as a straight line, then distance to cloud$=63.0648\sqrt{2}=89.18624$ kilometers.

If we regard the earth as a sphere, diameter 6370.946 kilometers, then $1°=111\frac{1}{9}$ kilometers. Therefore $34'\ 3.3''=63.0648$ kilometers.

$$\text{Distance to cloud}=\frac{6370.946\sin(34'\ 3.3'')}{\sin(44°\ 25'\ 56.7'')}=91.4594 \text{ kilometers.}$$

131. Proposed by SAUL EPSTEEN, Ph. D., The University of Chicago.

Find a power series for π^{nx} (n=any integer).

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$$\pi^{nx}=1+nx\log_e\pi+\frac{n^2x^2}{2!}(\log_e\pi)^2+\frac{n^3x^3}{3!}(\log_e\pi)^3+\ldots.$$

Now $\log_e \pi = \log_e[1+(\pi-1)] = (\pi-1) - \frac{1}{2}(\pi-1)^2 + \frac{1}{3}(\pi-1)^3 - \frac{1}{4}(\pi-1)^4 + \ldots = A$, suppose.

$\therefore \pi^{nx} = 1 + Anx + (Anx)^2/2! + (Anx)^3/3! + (Anx)^4/4! + \ldots$.

NOTE.—Frank Gilman, of Churchville, N. Y., sent in a solution of problem 129, which is based on Chauvenet's method. The method has the advantage of being more convenient in practice and lends itself more easily to logarithmic computation. We have not the space to publish the solution. Mr. Gilman gets as a result 57° 38′ 17″.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

166. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A teacher's monthly salary after $m=2$ increases of $p=20$ and $q=10\%$, is $\$M=\$120$. What was the original salary?

ALGEBRA.

177. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Solve $m^{2x}(m^2+1) = (m^{3x}+m^x)m$.

GEOMETRY.

200. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the locus of eight points of contact of the four common tangents of two concentric coaxial ellipses.

CALCULUS.

165. Proposed by CAPT. T. C. DICKSON, Ordnance Department United States Army, Washington.

Solve by integration, the differential equation

$$\frac{d^2\varphi}{dt^2} + \frac{A}{B}\left(\frac{d\varphi}{dt}\right)^2 - \frac{C}{B} = 0, \text{ in which:}$$

$$A = 1{,}103{,}430{,}032.196\sin\varphi\cos\varphi - 38{,}579{,}566.1706\sin^2\varphi + 38{,}575{,}641.7961\cos^2\varphi - 310.6332\cos\varphi + 204.6506\sin\varphi + 17.6818M\cos\varphi\sin\varphi + .4082M\sin^2\varphi - .4117M\cos^2\varphi - .3117M\sin\varphi + .0061M\cos\varphi,$$

$$B = 6382.5395\sin\varphi\cos\varphi + 59{,}363.1172\sin^2\varphi - .0095M\sin^2\varphi - 204.65\cos\varphi - -310.6332\sin\varphi - .8199M\cos\varphi\sin\varphi - .0095M\sin^2\varphi - 17.6904M\cos^2\varphi + .0061M\sin\varphi + .3117M\cos\varphi - 1310.866,$$

$C=\sin\varphi(16{,}209.4583-.0029254M)+\cos\varphi(101{,}111.3767-.12678\ M)-1080.2307+.001874M.$

For practical application, it is desired to find the mass M which is required to rotate certain known parts of a machine through the angle from $\varphi=13°$ to $\varphi=88°$ in the time $t=2''$.

MECHANICS.

155. Proposed by M. E. GRABER, Graduate Student, Heidelberg University, Tiffin, Ohio.

A parabolic curve is placed in a vertical plane with its axis vertical and vertex downwards, and inside of it, and against a peg in the focus, and against the concave arc, a smooth uniform and heavy beam rests. Find the position of equilibrium.

156. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, Eng.

Three perfectly elastic particles start from the cusp of a smooth cycloid (axis vertical, vertex down) at intervals of t seconds. How long will it be to the nth collision?

DIOPHANTINE ANALYSIS.

115. Proposed by LON C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Required the least three square integral numbers the difference between the sum of every two of them and the third shall be a square number.

AVERAGE AND PROBABILITY.

142. Proposed by ARTEMAS MARTIN, A. M., Ph. D., LL. D., Washington, D. C.

Two points are taken at random in the arc of a semi-circle, and a third point anywhere in its base. Find the probability that the triangle formed by joining them is acute.

[Unsolved Problem 9955, *Educational Times*, London.]

MISCELLANEOUS.

137. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

The first transvectant of the binary cubic and its second transvectant is the *cubico-variant* of the binary cubic.

NOTES.

Mr. Joseph Larmer, Fellow of St. John's College, Cambridge, has been elected Lucasian professor of mathematics to succeed the late Sir George Gabriel Stokes. D.

Dr. Arnold Emch, heretofore Assistant Professor of Pure and Applied Mathematics in the University of Colorado, has been promoted by the Regents to

a full Professorship of Graphics and Mathematics. Dr. Emch first came to the University in the spring of 1900. The following fall he was made an Assistant in Mathematics, and before the year closed he was elected Assistant Professor of Pure and Applied Mathematics. His recent promotion to a full Professorship brings Dr. Emch next to Professor DeLong in the Mathematical Department, and is a well-deserved recognition of his ability, faithfulness, and teaching power.

BOOKS.

Plane Geometry by the Suggestive Method. By John A. Avery, Head of the Mathematics Department, English High School, Somerville, Mass. 8vo. Board, vi+122 pages. Boston: Benj. H. Sanborn & Co.

In this work, the demonstrations are not given in full, but are outlined by means of hints or suggestions. No figures are given but space is left under each theorem so that the pupil can fill in the figure for himself.

The book should prove valuable in the hands of teachers having time to give personal attention to the work of each pupil. F.

A Text-Book of Field Astronomy for Engineers. By George C. Comstock, Director of the Washburn Observatory, Professor of Astronomy in the University of Wisconsin. 8vo. Cloth, x+202 pages. Illustrated. Price $2.50. New York: John Wiley & Sons.

This work is designed chiefly for students in technical schools, where the work in Astronomy is usually a part of a course of technical and professional training of students who have no purpose to become astronomers. Such students are often ignorant of many important problems in Astronomy which are of great value to them in practical life. In this work are considered such problems which, in the opinion of the author, are most useful, among them being:

Rough Determinations of Latitude from altitude; Time and azimuth from single altitude; approximate Determinations circum-meridian altitudes for altitude; Time for single-altitude, etc. F.

The Constructive Development of Group Theoy. By B. S. Easton. Publications of the University of Pennsylvania, Mathematics, No. 2. iv+89 pages. Cloth. Price, 75 cents.

This monograph presents in continuous form, but omitting all proofs, the main concepts and results of abstract and substitution group theory. While the theory of linear groups is expressly excluded, some of its results are given under "systems of simple groups," pp. 83–84. An exhaustive bibliography occupies only 34 pages. The treatise proper occupies 39 pages, exclusive of the 8 pages of tables. A systematic use of abbreviations for titles and journals has enabled the author to give a vast number of references in so short a space. The monograph will appeal both to the beginner and to the specialist in group-theory. A technical review, noting some minor corrections, has been offered to the Bulletin of the American Mathematical Society. D.

A Mathematical Solution Book. Fourth Edition, revised and enlarged. By B. F. Finkel, Professor of Mathematics and Physics in Drury College. Cloth, xvi+549 pages. Price, $2.00. Springfield: Kibler & Co.

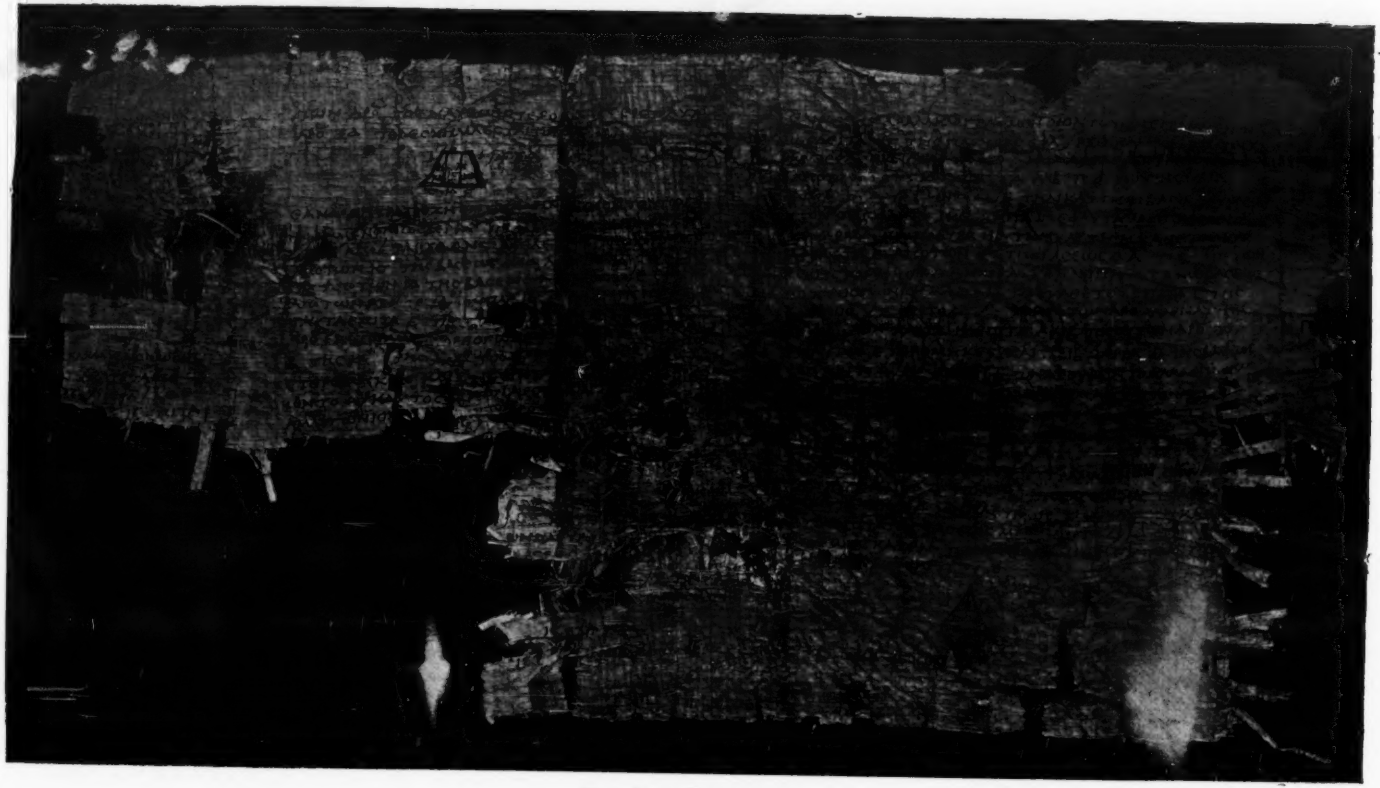
The need for such a text in English is evinced by its cordial reception, the book having passed through four editions in a comparatively short time. It aims to give "a solution of every problem presenting anything peculiar and of those problems which go the rounds of the conntry." While devoted chiefly to the essentials of arithmetic, algebra and geometry, it gives an abundance of interesting material not usually accessible to the teacher or student of elementary mathematics, and hence should prove a stimulus towards a wider knowledge of these subjects, It is aimed to supplement the usual texts. Occasionally a higher subject is drawn upon in the proofs; circulating decimals are treated by use of the algebraic theory of infinite geometrical progressions; the formulae in interest and annuities are obtained by algebraic methods; many of the formula in mensuration (after the 25 pages of elementary treatment) are established with the aid of Calculus. In the definitions of mathematical terms, the Greek and Latin derivations are given. The subject of higher plane curves receives considerable attention. As an exception to the usual care in the matter of definitions, that of a regular solid (p. 343) seems indaequate. The aggregate of two regular tetrahedrons, forming a dihedron, has its faces all equal, but is not a regular solid. Again, tetrahedron, octahedron, etc., are used by the author to denote regular solids only. As an introduction to geometry, a chapter on logic and logical fallacies is aptly inserted. The book surely merits the success it is receiving. D.

Niedere Zahlentheorie. Part I. By Dr. Paul Bachmann. B. G. Teubner, Liepzig. 1902. x + 402 pages.

This is the first of two volumes on elementary number theory written for Teubner's extensive collection of text-books on higher mathematics. It doesn't conflict seriously with Bachmann's ambitious series of volumes giving a comprehensive exposition of the theory of numbers, but rather forms a supplementary volume. An interesting feature is the exhaustive treatment of the known proofs by elementary number theory of the quadratic reciprocity law and the inter-relations of these proofs. While it is not as well adapted for a brief introduction to number theory as many of the current texts, it will appeal to the student wishing a complete account of the results and methods of the elementary parts of the theory of numbers. D.

Liniengeometrie mit Anwendungen. Volume I. By Dr. Konrad Zindler, Professor in the University of Innsbruck. Liepzig: G. T. Görshen, 1902. 8vo. viii+380 pages. 87 figures.

This text-book, containing numerous exercises with hints for their solution, forms volume 34 of Schubert's collection of texts. It is a systematic exposition of Linegeometry, employing both synthetic and analytic methods. The present (first) volume treats chiefly of linear complexes and congruences of lines, together with applications to the theory of motion, mechanics, and graphic statics. The second volume will treat algebraic line configurations of higher than the first degree. D.



THE AYER PAPYRUS. From the *American Journal of Philology*, Vol. X.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. X. MAY, 1903. No. 5.

ON THE PRODUCT OF AN ALTERNANT BY A SYMMETRIC FUNCTION.

By DR. W. E. TAYLOR, Associate Professor of Mathematics, Syracuse University.

As is well known, the product of the simple alternant $| a_1^0\ a_2^1\ a_3^2\ a_4^3 \ldots |$ and a symmetric function of $a_1, a_2, a_3, \ldots$ is an aggregate of alternants. When the symmetric function is of the form $\Sigma\, a_1 a_2 a_3 \ldots a_r$, the product is a single alternant differing from the original in having each of the last exponents increased by unity. In the general case, the mode of obtaining the aggregate is a fairly simple problem. The problem of obtaining the coefficient of a given alternant without having at the same time to obtain the coefficients of all the alternants in the aggregate is a problem not so simple. As a partial solution of this problem Dr. Muir has shown how the coefficient of one term of the aggregate may be obtained independently of the others.

The object of this paper is to extend the general problem started by Muir.

It is apparent that if we have a table giving the coefficients of all alternants in the product, (1)

$$| 0123\ldots(n-1) | \left(\sum a_1 a_2 a_3 \ldots a_{j_1}\right)^{i_1} \left(\sum a_1 a_2 a_3 \ldots a_{j_2}\right)^{i_2} \cdots \left(\sum a_1 a_2 a_3 \ldots a_{j_t}\right)^{i_k}$$

where $0 \leqq i_k \leqq n$, $1 \leqq j_k \leqq n$; $i_1 j_1 + i_2 j_2 + \ldots + i_k j_k = t \leqq n$; and where $i_1 + i_2 + i_3 + \ldots i_k = k$, we can by taking the proper multiples* of

$$(2)\quad \left(\sum a_1 a_2 a_3 \ldots a_{j_1}\right)^{i_1} \left(\sum a_1 a_2 a_3 \ldots a_{j_2}\right)^{i_2} \cdots \left(\sum a_1 a_2 a_3 \ldots a_{j_k}\right)^{i_k}$$

form any symmetric function S, and hence from such a table get the coefficients of any alternant in the product $| 0\ 1\ 2\ 3 \ldots (n-1) |\ S$.

*These multiples are found from the table of symmetric functions of weight t.

The difference equation for the coefficients of all alternants in the product (1) (where $i_1=1, j_1\geqq\frac{1}{2}n$) is obtained and this is used to construct tables for certain cases.

1. Let $|\,0\,1\,2\,3\ldots\ldots r_1s_1\ldots\ldots r_2s_2\ldots\ldots r_gs_g\ldots\ldots r_ks_k\,|$ denote the alternant wherein the numbers are consecutive except at the k points r_1s_1, r_2s_2, ..., r_ks_k,—that is, the numbers from 0 to r_1, s_1 to r_2, s_2 to r_3, and so on are consecutive; but r_1s_1, r_2s_2, etc., may differ by more than unity.

Let us denote the coefficient of this alternant in the product

$$|\,0123\ldots\ldots(n-1)\,|\,\Big(\sum a_1a_2a_3\ldots\ldots a_{j_1}\Big)\Big(\sum a_1a_2a_3\ldots\ldots a_{j_2}\Big)^{i_2}\ldots\ldots\Big(\sum a_1a_2a_3\ldots\ldots a_{j_h}\Big)^{i_h}$$

by

$$C^t\begin{Bmatrix} s_1 & s_2 & s_3\ldots\ldots s_g\ldots\ldots s_k \\ 0 & 0 & 0\ldots\ldots 0\ldots\ldots 0 \end{Bmatrix} \tag{3}$$

and the coefficient of

$$|\,0\,1\,2\,3\ldots\ldots r_1(s_1-1)\ldots\ldots(s_1+\alpha-1)(s_1+\alpha)\ldots\ldots r_2(s_2-1)s_2\ldots\ldots(s_2+\beta-2)(s_2+\beta)\ldots\ldots$$
$$\ldots\ldots r_g(s_g-1)s_g\ldots\ldots(s_g+\gamma-2)\ldots\ldots r_k(s_k-1)\,|\ \text{ in the product}$$

$$|\,0123\ldots(n-1)\,|\,\Big(\sum a_1a_2a_3\ldots a_{j_1}\Big)\Big(\sum a_1a_2a_3\ldots a_{j_2}\Big)^{i_2}\ldots\Big(\sum a_1a_2a_3\ldots a_{j_h}\Big)^{i_h-1}$$

by

$$C^{t-j_h}\begin{Bmatrix} s_1 & s_2\ldots\ldots s_g\ldots\ldots s_k \\ \alpha & \beta\ \ldots\ldots\gamma\ \ldots\ldots\kappa \end{Bmatrix},$$

where γ placed below s_g denotes that γ consecutive numbers beginning with s_g are decreased by unity.

2. Now it is easily seen that that the product

$$|\,0123\ldots\ldots(n-1)\,|\,\Big(\sum a_1a_2a_3\ldots\ldots a_{j_1}\Big)\Big(\sum a_1a_2a_3\ldots\ldots a_{j_2}\Big)^{i_2}\ldots\Big(\sum a_1a_2a_3\ldots\ldots a_{j_h}\Big)^{i_h-1}$$

$$=\ldots\ldots+\sum_0^{j_h} a\beta\ldots\ldots\kappa\ C^{t-j_h}\begin{Bmatrix} s_1 & s_2\ldots\ldots s_k \\ a & \beta\ \ldots\ldots\kappa \end{Bmatrix}\,|\,0\,1\,2\,3\ldots\ldots$$

$$\ldots\ldots r_1(s_1-1)s_1(s_1+1)\ldots\ldots(s_1+\alpha-2)(s_1+\alpha)\ldots\ldots r_g(s_g-1)s_g\ldots\ldots(s_g+\gamma-2)(s_g+\gamma)\ldots\ldots$$

$\ldots\ldots r_k(s_k-1)\,|\,+$ other terms, where $\alpha+\beta+\ldots\ldots+\kappa=j_h$; κ cannot of course be greater than 1.

Only those terms are written which on multiplying both sides by $\Sigma a_1a_2a_3\ldots a_{j_h}$ can give rise to

$$|\,0\,1\,2\,3\ldots\ldots r_1s_1\ldots\ldots r_2s_2\ldots\ldots r_gs_g\ldots\ldots r_ks_k\,|\ ;$$

but the coefficient of this term in the product is (3). Hence this coefficient must be equal to the sum of the coefficients of the above terms, that is